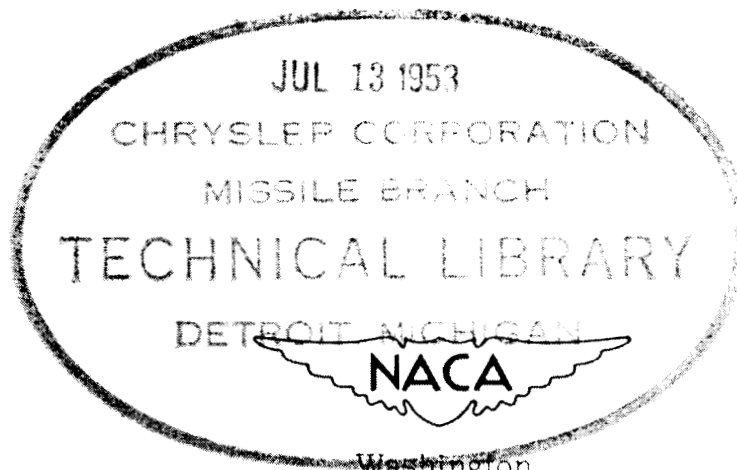# NATIONAL ADVISORY COMMITTEE
FOR AERONAUTICS

TECHNICAL MEMORANDUM 1359

AIR-WATER ANALOGY AND THE STUDY OF HYDRAULIC MODELS

By Giulio Supino

Translation of "La Similitudine Aria-Acqua e lo Studio dei Modelli
Idraulici." L'Energia Elettrica (Milano), vol. XXVIII,
no. 11, Nov. 1951

NACA

Washington

July 1953

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM 1359

AIR-WATER ANALOGY AND THE STUDY OF HYDRAULIC MODELS[*][1]

By Giulio Supino

SUMMARY

The author first sets forth some observations about the theory of
models. Then he establishes certain general criteria for the construc-
tion of dynamically similar models in water and in air, through refer-
ence to the perfect fluid equations and to the ones pertaining to viscous
flow. It is, in addition, pointed out that there are more cases in which
the analogy is possible than is commonly supposed.

AIR-WATER ANALOGY AND THE STUDY OF HYDRAULIC MODELS

1. The use in model experiments of a fluid which is different from
that of the original one has been known for some time. One is reminded
in this connection that tests are made in water of bodies which are
designed to operate in air (dirigibles, airplanes, railway cars, etc.),
and the first investigations in this direction were carried out by G. A.
Crocco in 1906[2]; and it is recalled that tests are made in air (that is,
in wind tunnels) on configurations which are designed to operate in

---

[*]"La Similitudine Aria-Acqua e lo Studio dei Modelli Idraulici."
L'Energia Elettrica (Milano), vol. XXVIII, no. 11, Nov. 1951.

[1]Theoretical background for experiments now going on at the
Hydraulics Laboratory of the University of Bologna, sponsored by the
ANIDEL (National Association of Electric Power Companies).

[2]In particular, one should consult:
    Crocco: Dynamics of Dirigible Balloons (Dinamica degli Aerostati
            Dirigibili), Bulletin of the Italian Aeronautics Society
            (Bollettino della Società Aeronautica Italiana),
            April 1907.
    B. Finzi Contini: Concerning an Experimental Method for the Study
            of Car Models (Su di un Metodo Sperimentale per lo Studio
            dei Modelli di Veicoli), The Engineer (L'Ingegnere),
            1935.
    G. Corbellini: The engineering of rail transportation (Tecnica
            dei Trasporti su Rotaia). Rome, 1941, pp. 210-232.

water, or are otherwise subjected to hydraulic loads[3]. Within this frame
of reference the Hydraulics Laboratory of the University of Bologna has
in progress the setting up of the necessary mechanical devices for the
carrying out of certain research work, and this paper, in the form of a
preview, is intended to present some facets of the possibilities inherent
in experimentation that can be conducted with these two fluids.

2. Some considerations of a general nature are to be discussed
first of all. The theory of mechanical similitude can be developed from
two different points of view. The first of these ways of looking at the
question is of an inductive sort, and it is based on the π-theorem (of
Vaschy-Riabuscinski-Buckingham), while the second viewpoint is deductive
in nature and stems from the differential equations which govern the
phenomenon.

Since the import of the two methods of approach each differs from
the other, it is worth while to review them briefly with respect, how-
ever, merely to the hydraulic problem. The expression of the π-theorem
is then as follows:

"Let us assume that the following physical quantities are involved
in the hydraulic phenomena under consideration:

ρ   density of the fluid

l linear dimension which characterizes the body whose behavior
                is to be scrutinized

U   average velocity of the fluid

f   force (drag of the moving fluid on a given length of tube,
                thrust of a jet against a restraining wall or of a stream
                against an immersed body, drag of the fluid resisting the
                motion of an immersed or floating body)

---

[3]Tests made in air, in a pressure tunnel, to study liquid flows may
be found in:
G. Remenieras: Prédetermination de la perte de charge dans une
             canalisation d'eau sous pression à partir de celle
             mesurée sur le même canalisation parcourue par de l'air.
             Third Congress of the AIRH, Grenoble 1949, vol. III,
             no. 2.
P. Bourguignon: Prédetermination de la perte de charge dans une
             canalisation d'eau sous pression à partir de celle
             mesurée sur le même canalisation parcourue par de
             l'air. Contrôle de la validité de le méthode sur la
             galerie d'amenée R.G.de l'usine Hydro-electrique du Pont-
             Escoffier. Ibid. III.9.

g   acceleration of gravity

μ   absolute viscosity of the fluid

r_s nondimensional parameter or group of parameters which char-
           acterize the roughness of the walls

r_f nondimensional parameter or group of parameters which char-
           acterize the shape of the body

The connection between these quantities is to be expressed by a
function of the form

$$\varphi\left(\rho,l,U,f,g,\mu,r_s,r_f\right) = 0 \tag{1}$$

Because this function is not to undergo any modification when the
dimensional units are altered, then the unknown quantities will have to
appear regrouped in such a way as to constitute pure numbers. Now all
the distinct pure numbers which it is possible to construct through use
of the quantities ρ, l, U, f, g, μ (of course r_s and r_f are
already pure numbers) are the following

$$N_1 = \frac{\rho l^2 U^2}{f}$$

$$N_2 = \frac{U^2}{gl} \qquad \text{(Froude's Number)}$$

$$N_3 = \frac{\rho l U}{\mu} \qquad \text{(Reynolds' Number)}$$

Thus in place of equation (1) one may write

$$\varphi_1\left(N_1,N_2,N_3,r_f,r_s\right) = 0 \tag{2}$$

This relationship is often written in a more simple manner as a
result of the fact that it is possible to rearrange it so that one of

the unknown quantities is given explicitly; for example, in the case of the force f, we have that the relationship

$$f = \rho l^2 U^2 \psi\left(N_2, N_3, r_f, r_s\right) \tag{2'}$$

holds."

Suppose now that it is required that some hydraulic phenomenon is to be studied by means of a model. In addition to the geometric conditions of similitude

$$\left.\begin{array}{l} r_s = r_s' \\ r_f = r_f' \end{array}\right\} \tag{3}$$

which must be satisfied (where primes refer to the quantities related to the model), it is also necessary to specify the further conditions of mechanical similarity given by

$$\left.\begin{array}{l} \rho\, \dfrac{lU}{\mu} = \rho'\, \dfrac{l'U'}{\mu'} \\[2ex] \dfrac{U^2}{gl} = \dfrac{U'^2}{gl'} \end{array}\right\} \tag{4}$$

and

$$\rho\, \frac{l^2 U^2}{f} = \rho'\, \frac{l'^2 U'^2}{f'}$$

If the density of the fluid is ρ and its viscosity μ in the "basic" or original system, and if the model is to be tested in a fluid for which the density and viscosity are to be ρ' and μ', respectively, the preceding conditions (4) constitute a system of relationships in the unknowns

$$\frac{l}{l'}, \quad \frac{U}{U'}, \quad \text{and} \quad \frac{f}{f'}$$

Let us denote by $\nu = \dfrac{\mu}{\rho}$ the kinematic viscosity of the basic fluid, and likewise let ν' denote the same quantity for the model system, then by means of equation (4) the following are obtained

$$\left.\begin{array}{l}
\dfrac{l}{l'} = \left(\dfrac{\nu}{\nu'}\right)^{2/3} = k^{2/3} \\[2ex]
\dfrac{U}{U'} = \left(\dfrac{\nu}{\nu'}\right)^{1/3} = k^{1/3} \\[2ex]
\dfrac{f}{f'} = \left(\dfrac{\rho}{\rho'}\right)\left(\dfrac{\nu}{\nu'}\right)^{2} = \left(\dfrac{\rho}{\rho'}\right)k^{2}
\end{array}\right\} \qquad (5)$$

where k has been set in place of $\dfrac{\nu}{\nu'}$.

In the case of the other mechanical parameters the following reduction ratios apply

time $\qquad\qquad\qquad \dfrac{t}{t'} = k^{1/3}$

acceleration $\qquad\quad \dfrac{a}{a'} = 1$

volume flow $\qquad\quad \dfrac{Q}{Q'} = k^{5/3}$

pressure $\qquad\qquad \dfrac{p}{p'} = \left(\dfrac{\rho}{\rho'}\right)k^{2/3}$

static head $\qquad\quad\; \dfrac{h}{h'} = k^{2/3}$

momentum $\qquad\quad \left(\dfrac{m}{m'}\right)\left(\dfrac{U}{U'}\right) = \left(\dfrac{\rho}{\rho'}\right)k^{7/3}$

work $\qquad\qquad\qquad \dfrac{L}{L'} = \left(\dfrac{\rho}{\rho'}\right)k^{8/3}$

power $\qquad\qquad\quad \dfrac{P}{P'} = \left(\dfrac{\rho}{\rho'}\right)k^{7/3}$

angular velocity $\quad \dfrac{\omega}{\omega'} = k^{-1/3}$

If air is selected as the fluid to be used in the study of a
hydraulic phenomenon, the value of the ratio between the kinematic
viscosities of water and air, ν/ν', cannot range out of too extensive
bounds. In fact, for ordinary temperatures the value of this ratio, as
a function of temperature in the two fluids, varies from a maximum of
0.13 at 0° C to a minimum of 0.05 at 30° C, as may be verified by refer-
ence to figure 1.*

On the other hand, if one uses, as a basis of comparison, the kine-
matic viscosity of water at 15°, the ratio between the viscosities can
at most vary from the value

$$\frac{\nu \text{ of water at } 15° \text{ C}}{\nu' \text{ of air at } 0° \text{ C}} = \frac{1.14 \times 10^{-6}}{1.36 \times 10^{-5}} = 0.0838$$

for air at a temperature of 0° C to the value

$$\frac{\nu \text{ of water at } 15° \text{ C}}{\nu' \text{ of air at } 30° \text{ C}} = \frac{1.14 \times 10^{-6}}{1.64 \times 10^{-5}} = 0.0695$$

for air at a temperature of 30° C.

By use of equation (5), it is seen that the corresponding values of
the ratio l/l' are, respectively, 0.187 and 0.169, to conform with these
just-mentioned values of the ratio ν/ν'. In the first case, that is,
the scale of the model will have to be 5.35 times larger than that of
the original, while in the second case it will have to be 5.90 times
greater.

It is clear that if the average temperature of the flow in which
the original body is to operate as a water model is different from this
15°, other values for the l/l' ratio will be obtained for the range of
air temperatures under which the experimentation is done; however, these
values will not deviate greatly from the limiting values just adduced
above.

One may thus conclude that it will be useful to employ dimensional
experiments with models in air to large scale whenever the original

---

*The NACA reviewer suggests that the exponents of 10 in the values
of kinematic viscosity for water and air should be 5 and 4, respectively;
however, the ratios of these values are unaffected.

setup in water is of such small size that it would not permit an accurate examination of its behavior.

On this account dimensional experiments can be employed for the study of flow from vents, from small valves, etc. where the influence of gravity and viscosity of the fluid can be taken into account at the same time.

This mode of approach to the subject of mechanical similitude has, as mentioned earlier, an inductive aspect, that is, it is substantiated only provided that the initial hypothesis is true that the hydraulic phenomenon under examination depends only on the physical quantities enumerated in equation (1).

When all these quantities (and only these) come into play in more diverse theories, then I have already demonstrated earlier that the analogy between model and original is common to all the theories thus constructed, and thus the model test and the transfer from model to original lead to a result which, in a certain sense, does not depend upon these theories[4]. On the other hand equations (3) and (4) are not amenable to any further polishing up; i.e., once the dependence of the function φ_1 on N_1, N_2 and N_3, r_f and r_s is established, no further importance can be attached to one or the other of these numbers, nor can any criterion be set up for preferring one over the other. The elimination of the quantities which may be considered to be superfluous has to come about through the initial supposition that the phenomenon does not depend on them. If one or more quantities are cut out then also one or more of the pure numbers among the group found previously will be lacking. For example, if one is concerned with a perfect fluid (for which $\mu = 0$), then the Reynolds number would be missing.

If the liquid is to be uninfluenced by the force of gravity, then the Froude number would be missing and if one should like to forget about the density, ρ, then the numbers N_1 and N_3 would be lacking. In this case, the number $\overline{N}_1$, defined as $\overline{N}_1 = \dfrac{\mu l U}{f}$ would be inserted in place of N_1 and N_3, in addition to having N_2 still involved.

3. It now remains to ask whether it is possible to extend the range of usefulness of the air experiments by reducing the dimensions of the model. The answer is in the affirmative, because it is not actually

---

[4]G. Supino: On dimensional analysis and the theory of models (Su l'analisi dimensionale e la teoria dei modelli) Two notes in the "Rendiconti della Accademia Nazionale dei Lincei", Sept. 1, 1947.

necessary to keep in the Froude number, and upon dropping the pure number N_2, a much greater liberty of choice is vouchsafed. But in order to study this, it is necessary to study the hydraulic analogy from the deductive point of view, as applied to a particular one of the analogies.

Let us focus our attention, for example, on the equations describing the behavior of an incompressible viscous liquid (that is, on the Navier equations).

These Navier equations are constituted of:

(a) The equations of motion:

In addition to the symbols already defined in section 2, the following nomenclature is introduced at this point. The local components of the forces acting on a unit mass are denoted as X, Y, Z which are components directed along the corresponding coordinate axes, respectively; p = pressure, μ = viscosity of the liquid, ρ = density, and $\frac{du}{dt}$, $\frac{dv}{dt}$, and $\frac{dw}{dt}$ are the "material" derivatives of the velocity components, that is to say, they are components of the acceleration of a little particle of liquid which, at the time t, is located at the point with coordinates x, y, and z. The symbol Δ stands for the operator

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Then the equation of motion is

$$\left.\begin{array}{l} \rho X - \dfrac{\partial p}{\partial x} + \mu \Delta u = \rho \dfrac{du}{dt} \\[2mm] \rho Y - \dfrac{\partial p}{\partial y} + \mu \Delta v = \rho \dfrac{dv}{dt} \\[2mm] \rho Z - \dfrac{\partial p}{\partial z} + \mu \Delta w = \rho \dfrac{dw}{dt} \end{array}\right\} \tag{6}$$

(b) The equation of continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{7}$$

(c) The boundary conditions

In reference to these, there are no uncertainties introduced here in the case of flow along walls, because it is obvious that the liquid sticks to them, and thus its velocity is equal to that of the wall, and also it is seen that the protuberances are reduced in the same proportion as required by the geometric proportionality factor. The question of "free surfaces" will, on the other hand, be considered later.

It may be deduced directly from equations (6), that is to say, without reliance upon the π-theorem) what the conditions are for similitude to hold in the gravitational field.

Let us consider, therefore, the third of equations (6), applying to the original setup. Let the z-axis be directed upwards, whence $Z = -g$; then we have

$$-g - \frac{1}{\rho} \frac{\partial p}{\partial z} + \frac{\mu}{\rho} \Delta w = \frac{dw}{dt}$$

Let p', ρ', z', etc. be the magnitudes related to the model conditions; then in this case we have

$$-g - \frac{1}{\rho'} \frac{\partial p'}{\partial z'} + \frac{\mu'}{\rho'} \Delta w' = \frac{dw'}{dt'}$$

But it is true that

$$p' = \pi p, \quad \rho' = \delta \rho, \qquad \nu' = \frac{\nu}{k}$$

$$z' = \lambda z, \quad w' = \lambda \tau^{-1} w, \quad \Delta w' = \lambda^{-1} \tau^{-1} \Delta w$$

so that upon comparison of the equation relating to the original conditions to that relative to the model, it follows that

$$\frac{\delta \lambda}{\pi} = k \lambda \tau = \frac{1}{\lambda \tau^{-2}} = 1$$

because g remains unchanged in the two cases.

The condition that $\lambda\tau^{-2} = 1$ is equivalent to the one denoted by $\frac{u^2}{gl} = \text{const.}$ (Froude's no. required to remain unchanged).

The condition that $k\lambda\tau = 1$ when combined with the preceding condition leads one to write

$$k\lambda\tau\left(\lambda\tau^{-2}\right) = k\lambda^2\tau^{-1} = 1$$

and this corresponds to the Reynolds number condition.

Finally, the condition $\frac{\delta\lambda}{\pi} = 1$ when combined with the condition $\lambda\tau^{-2} = 1$ and by making use of the relationship that $\pi = f\lambda^{-2}$ leads one to write

$$\frac{\delta\lambda^3}{f}\left(\lambda\tau^{-2}\right) = \frac{\delta\lambda^4\tau^{-2}}{f} = 1$$

that is to say, we have $\rho\,\frac{l^2 U^2}{f} = \text{const.}$

It must be remembered that the law of geometric similtiude must be upheld, and thus equations (3) are valid. Consequently we are brought back to equations (4). It is evident that equation (7) and the conditions at the wall do not impose any other restrictions.

4. The preceding result does not inform us of anything that could not also be derived from the π-theorem. But considering that this time we have within our grasp the very equations with respect to which we want to deduce the law of similitude, then it is thus possible to intimate that a deeper study of these equations leads to a reduction, in certain cases, of the number of imposed conditions. A very simple trick way of doing this is as follows, which is a well-known device, at least theoretically (compare, for example, p. 34 of Pistolesi's Aerodinamica).

Let us return to examination of equation (6). Let the pressure p be divided up into two portions so that $p = p_1 + p_2$. After doing this, we may write the equations of motion which apply in a gravity affected field as

$$-\frac{\partial(p_1 + p_2)}{\partial x} + \mu\,\Delta u = \rho\,\frac{du}{dt}$$

$$-\frac{\partial(p_1 + p_2)}{\partial y} + \mu\,\Delta v = \rho\,\frac{dv}{dt} \qquad\qquad (8)$$

$$-\gamma - \frac{\partial(p_1 + p_2)}{\partial z} + \mu\,\Delta w = \rho\,\frac{dw}{dt}$$

If we now substitute for p_1 the value $p_1 = -\gamma z$, then we shall wind up with nothing more than the equations

$$-\frac{\partial p_2}{\partial x} + \mu\,\Delta u = \rho\,\frac{du}{dt}$$

$$-\frac{\partial p_2}{\partial y} + \mu\,\Delta v = \rho\,\frac{dv}{dt} \qquad\qquad (9)$$

$$-\frac{\partial p_2}{\partial z} + \mu\,\Delta w = \rho\,\frac{dw}{dt}$$

and the law of similitude can be deduced by stipulating in these relationships that it must hold true that

$$\frac{\delta\lambda}{\pi} = k\lambda\tau = \frac{1}{\lambda\tau^{-2}}$$

but without the qualification that these three ratios must necessarily be equal to unity.

The solution that comes most readily to mind consists therefore in selecting the Froude number in a purely arbitrary manner and in demanding that the following two relationships hold

$$\frac{\delta\lambda\left(\lambda\tau^{-2}\right)}{\pi} = 1 \quad \text{and} \quad k\lambda^2\tau^{-1} = 1$$

that is to say, that the N_1 and N_3 parameters should remain constant; so that one may write

$$\varphi_1\left(N_1, N_3, r_s, r_f\right) = 0$$

If the function φ_1 can be cast into explicit form (let us suppose it is the drag force which we want to work with) then we get

$$f = \rho l^2 U^2 \psi\left(N_3, r_f, r_s\right)$$

which is just a way of saying that the drag force, f, turns out to be a function solely of the Reynolds number and of the shape parameters r_s and r_f [5].

Nevertheless the experimental facts will have to be handled in the following way. First carry out the tests of a hydraulic phenomenon, directly based on the Navier equations, by means of a model whose dimensions are to be obtained from those of the original configuration through

[5]It should be remarked that this trick also is of interest from a conceptual standpoint in that it serves to sever the drag measured in tubes from the Froude number. In fact, let us write down the expression for the change in the drag in progressing from one point in a tube to another as $f = \gamma i l^3$; then it follows that

$$i = \frac{u^2}{gl}\psi\left(N_3, r_f, r_s\right) = \frac{\lambda\left(N_3, r_s\right)}{D}\frac{u^2}{2g}$$

where the symbol $\lambda\left(N_3, r_s\right)$ denotes the nondimensional function which appears conventionally in all the formulae relating to drag in cylindrical tubes.

application of the premise that the Reynolds number is to remain constant. After doing this experimentation, the next step will be to subtract the value $p_1' = -\gamma'z'$ from the total gross pressure measured at each point of the model. It is worth pointing out that the value of p_1' is known at every point because the shape of the model is defined.

Thus one will have at hand the quantities $p_2' = p' - p_1'$, and from these values the law of similarity will permit the determination of p_2 [6]. The pressure existing at the corresponding point on the original body will be given by the relation: $p = p_2 - \gamma z$.

In all those cases where the value of the pressure is of no consequence to the experiment, no corrections of any sort are called for.

It may be further remarked that if the viscosity has but a minute influence on the hydraulic phenomenon under study, then the equations which govern the flow are those of a perfect fluid, and upon subjecting these equations to the same trick, one will arrive at the conclusion that the law of similarity is completely at our arbitrary disposal (provided that the pressures are computed by means of the correction factors presented above, after the similarity ratios have been selected and fixed).

5. Even when one takes into account the viscosity and stipulates therefore the constancy of the Reynolds number, the artifice being exploited here renders it possible for the dimensions of the model to be reduced from those applying to the original case. Set $k = \dfrac{\nu}{\nu'}$, $\lambda = \dfrac{l}{l'}$ (the symbol λ used here is the reciprocal of the one employed in the previous section) and let $\sigma = \dfrac{U}{U'}$.

When this is done it, of necessity, will follow that $\dfrac{\sigma\lambda}{k} = 1$. Now if k lies in the range $0.07 \leq k \leq 0.08$ and if λ is taken as 10, then it will be possible to obtain the desired result, namely, a reduction in the size of the model to 1/10 of the original, by stipulating that σ should equal $\dfrac{7}{1000}$ to $\dfrac{8}{1000}$; that is, by demanding that the model be tested at velocities of from 140 to 125 times those experienced by the original configuration.

6

From the ratio $\dfrac{p'}{p} = \pi = \delta\lambda^2\tau^{-2}$ it follows that $p_2 = \dfrac{p'}{\delta\lambda^2\pi^{-2}}$.

This condition is certainly quite severe, but it is much less so than the one which might have been imagined at first thought. Suffice it to say, in this regard, that velocities in excess of 100 m/sec can be reached and surpassed in aerodynamic wind tunnels[7].

On the other hand, when the Reynolds number has a large value it is not necessary to insist that the two Reynolds numbers applying to the original and to the model be definitely equal. It is sufficient to require that the model Reynolds number be merely large enough to render the drag independent of this N_3 parameter itself.

In contrast, the creation of a water analogy to the scale of 1 to 10 obliges one to seek (in order to obtain the same Reynolds number) a velocity which is ten times greater than in the original. But the difficulty inherent in reaching a fluid flow velocity of 7 to 8 m/sec in a laboratory model is much worse than that of trying to produce a velocity of 100 m/sec in a wind tunnel. Besides, the experiments on models operating in an air stream can be performed with very limited apparatus (inasmuch as such apparatus will have to withstand relatively weak forces) and thus, in addition, the dimensions can be enlarged. Such advantages will make the carrying out of the experiments a yet more simple task.

Finally, when one concentrates his attention solely on the inertial drag forces, the dimensional analogy may be carried out for the case of a perfect fluid field. In this case, regardless of whether one does or does not take into account the Froude number, one can take advantage of sizably larger changes in velocity by employing air-test models.

6. A single difficulty continues to bedevil the experimenter, and this blemish is not easily eliminated. This troublesome point arises in the case when any free surface or surfaces are present. The substantial difference which exists between the free surfaces present in the two cases can be explained, from a physical point of view, as follows: The jet discharge of the air stream occurs in air and the discharge of the water jet takes place likewise in air in the usual case, so that the density ρ is continuous across the free surface (while only the velocity is discontinuous there) for the former jet, but in the latter setup both the velocity as well as the density are discontinuous.

---

[7]Nevertheless it is important in the research we are doing to refrain from using velocities greater than about 80 to 90 m/sec in order to keep from surpassing by any appreciable amount the 25 percent-of-sound-velocity borderline. With regard to this subject the reader's attention is directed to a recent paper by J. W. Ball, entitled: Model Test Using Low-Velocity Air, Proceedings of the American Society of Civil Engineers, June 1951, Vol. 77, No. 76.

Pursuing this line of thought further it is seen that when the water jet discharge takes place in water (as for example, for flow across a completely submerged orifice) then there is no difference between the two cases, and the results derived as applying in air hold in addition for the case of the water flow, and vicé versa $(N_3$ being equal in the two cases).[8]

But in general, when talking about an air jet immersed in air, the free surface constitutes an interface which separates the flow into two distinct regions A and B; in the former the fluid is in motion, while in the latter it is at rest.

If we now restrict ourselves here to consideration of perfect fluids in irrotational steady flow, it follows that the actual pressure head is constant throughout the whole fluid, or

$$z + \frac{p}{\gamma} + \frac{v^2}{2g} = \text{const.}$$

(with the flow vector being assumed directed along the z-axis) and this relationship applies validly right on the interface which separates the two regions.

Now since, p, γ, and z are continuous across the surface while V_B is zero, then it follows that

$$V_A^2 = \text{a const.}$$

In other words, the velocity is constant on a free surface in steady flow (it being taken for granted that the free surface is interpreted as the interface between the air to air regions), and thus it follows in addition that the pressure head is also constant.

If one confines himself to examination of two-dimensional motion, then V is constant on the interface between the two regions, and in addition the stream function ψ is also constant[9].

---

[8]See, for example, the results of Zimm: Ueber die Strömungs Vorgänge im freien Luftstrahl. Forschungsarbeiten VDI, Heft 234, 1921, and those results obtained by Citrini concerning the spreading of a liquid jet in L'Energia Elettrica, 1946.

[9]The velocity V and the stream function ψ are constant in the two-dimensional case even if the motion is rotational.

It is quite different in the case of a free surface between air and water. For this type of flow, the quantities p, g, and z are again continuous across the interface, but γ is discontinuous. For this reason one may write that (for irrotational flow of a perfect fluid)

$$\frac{1}{2}\left(\rho_A V_A^2 - \rho_B V_B^2\right) + z\gamma_A - z\gamma_B = 0$$

and as a result of dividing through by γ_A and remembering that $V_B = 0$, it follows that

$$\frac{V_A^2}{2g} + z = z\,\frac{\gamma_B}{\gamma_A}$$

If one now drops out the term $\dfrac{\gamma_B}{\gamma_A}$ inasmuch as the specific weight of air is about 1/800 of that of water, then the simple result

$$\frac{V_A^2}{2g} + z = 0$$

holds.

Thus, in this case, now p = const on the interface; so that, whereas the pressure head is constant on the interface (or free surface) between air and air, the pressure is constant on the interface between water and air. The two cases become identical only when the free surface is a horizontal plane[10].

7. In the case of steady two-dimensional flows one gets ψ = const. in both situations. On this account, if one can set up an air-to-air model for a steady irrotational two-dimensional flow wherein the interface is in geometric analogy with the corresponding interface of the original water-to-air setup, even if only along a very small section of such an interface, and if the velocities are moreover in geometric similarity, then the analogy is vouchsafed to hold throughout the flow field. In

---

[10]If the velocity V_A is very large along the free surface, then the influence of the change in height z is very small; in this case also, therefore, the analogy between the air and water regimes of flow is approximately admissible.

fact, in this case the functions ψ and ϕ exactly correspond along the piece of the free surface. But once having the ψ-function given equal to the ϕ-function or having given the ψ-function and its normal derivative, thence, by means of a well-known theorem of Kirchhoff, the ψ-function is determined throughout the whole flow field[11].

As it has been remarked, this correspondence has to be established, however, on a free surface or on an element of surface immersed in the fluid - not on a fixed wall, where the condition of no-slip at the wall contravenes the irrotationality of the flow[12].

8. The conclusion from what has been set forth above is the following: The laws of hydraulic analogy can in general be established starting out from an inductive or a deductive origin as basis for consideration. In the former case one makes a list (hypothetically) of all, and only, those factors which are involved in the phenomenon. The similarity conditions are deduced then by means of the π-theorem, and these conditions are validated through means of other well-established theorems. In the second case (the deductive one) the physical factors which are involved in the phenomenon appear in the equations which govern the theory, and with these as a basis the law of similarity is formulated (without it being necessary to bring up the π-theorem in any shape or manner). These equations also serve to suggest some very useful tricks.

In the case of the mechanics of fluid flow these tricks have led us to establishment of the fact that experimentation performed in air on hydraulic phenomena has a quite wide field of application. Summarizing the results of sections 4, 5, 6, and 7 we can declare that it is possible to carry out experiments in air on hydraulic phenomena when one of the following conditions is conformed to:

1. The Froude and Reynolds numbers analogy is maintained (in other words the gravitational and viscous effects are taken into account), when a model is made which has a larger scale than that of the original.

2. The Reynolds number analogy is maintained during a reduction in scale (relegating to an insignificant role the influence of gravity or taking it into account by means of the artifice mentioned in section 4).

3. An analogy based on arbitrary ratios when the liquid can be considered to be a perfect fluid (taking into account the effect of gravity by means of the artifice mentioned in section 4).

---

[11]See Kirchhoff: Mechanik, pp. 187-188; and Almansi: Rendiconti della Acc. Naz.dei Lincei, Sept. 1, 1907, pp. 865-867.

[12]G. Supino: Sul Moto Irrotazionale dei Liquidi Viscosi, Rendiconti della Acc. Naz.dei Lincei, Serie VIII, Vol. VI, p. 615, et seq., Sept. 1, 1949.

The similarity relationship is applicable in all cases where free surfaces do not enter the problem. When these surfaces are present the cases in which similarity can be brought into play are discussed in section 6.

Translated by R. H. Cramer
Cornell Aeronautical Laboratory, Inc.
Buffalo, New York

Bibliographic Notes

The publications concerning dimensional analysis, similarity laws, and the theory of models are so numerous that it is not possible to list them all. We wish to record here merely those of most importance and those most easily obtained. Thus we first mention the work of

P. W. Bridgman, Dimensional Analysis, New Haven, 1932; also translated into German as Bridgman-Holt, Theorie der physikalischen Dimensionen, Leipsig, 1932.

In addition to this basic test we wish to add the papers of:

F. Marzolo, Alcune Considerazioni sui Modelli Idraulici, Giornale del Genio Civile, 1917.

G. DeMarchi, Omogeneità, Similitudine e Modelli Idraulici, Rendiconti del Seminario Mat. e Fisico di Milano, 1933.

E. Pistolesi, Omogeneità, Similitudine, Modelli, "Fondamenti Teorici" of the Atti del II Conv. di Mat. Applicata, 1940.

A quite extensive bibliography will be found in:

U. Puppini, Idraulica, Bologna, 1947, pp. 617-618.

While the less recent research work is cited and partly paraphrased in the book:

Hydraulic Laboratory Practice, New York, 1929.

Moreover one should consult the various notes of the late Prof. E. Foà, contained in the issues of L'Industria, from 1928 to 1929, and in Politecnico for the years 1929 to 1930 (these articles advance a point of view that is a little different from that brought out in sections 2 and 3).

Other works, in particular the most recent one of J. W. Ball, are referenced in the notes appended to the bottoms of the pages.
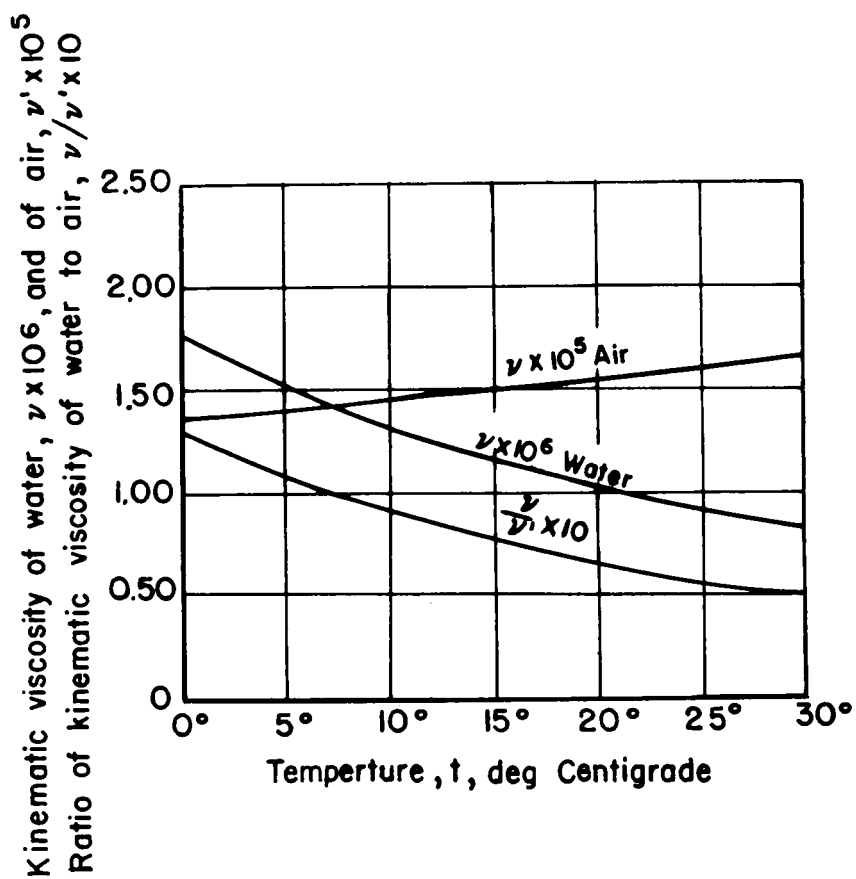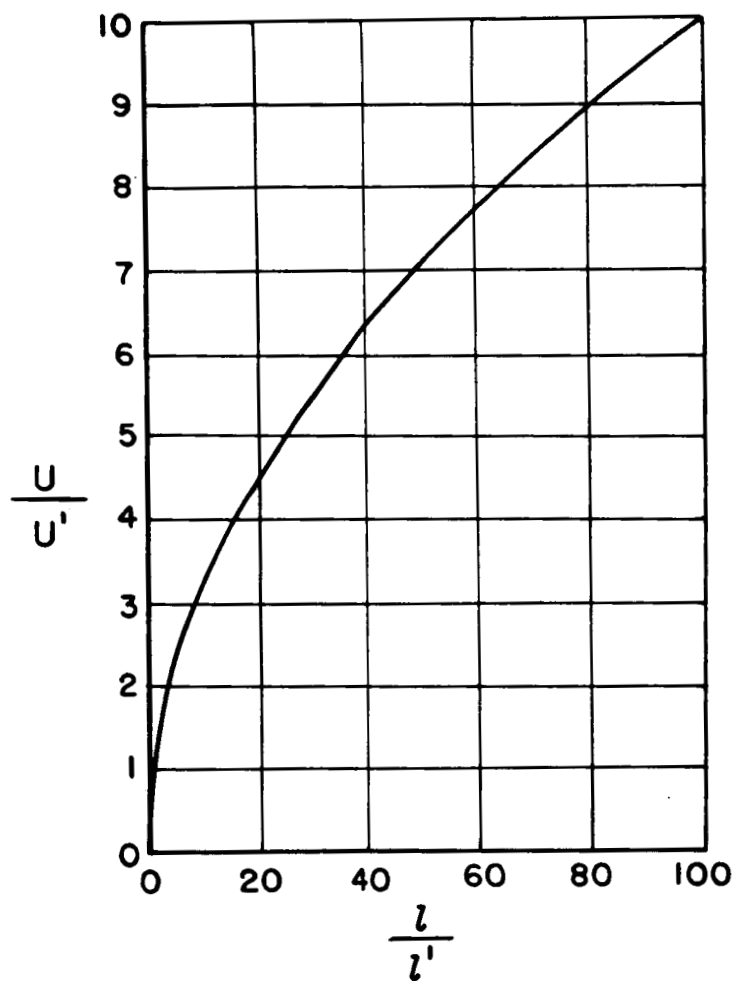
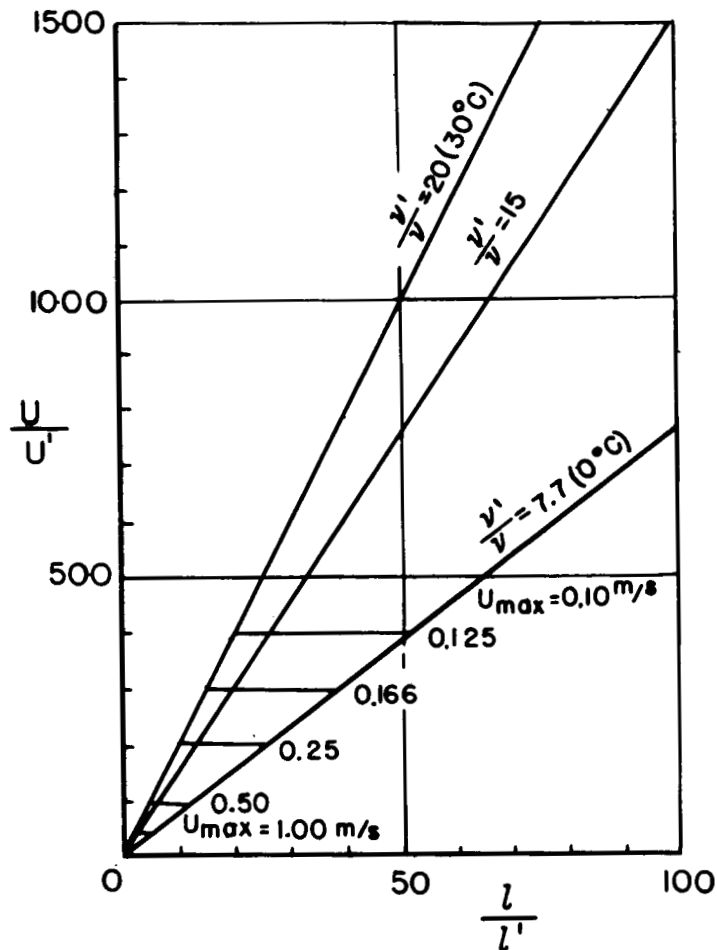Figure 1

Figure 2.- Froude number analogy.

Figure 3.- Reynolds number analogy $\dfrac{U\mathit{l}}{\nu} = \dfrac{U'\mathit{l}'}{\nu'}$. (The symbols differentiated by primes refer to the "air" conditions, while the unmarked ones relate to the water case. The U_{max} velocities are calculated to correspond to a U' value of 50 m.sec.)